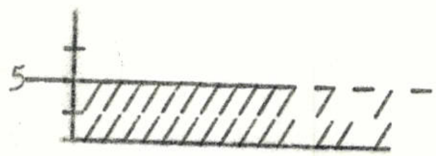
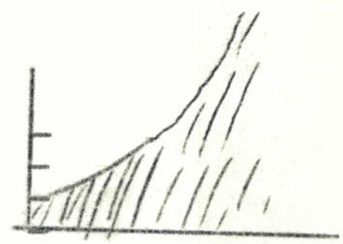
Calculus:
The Definite Integral

1. The definite integral has many uses, the simplest of which is probably area. Recall that in chapter 4 it was used in this way. Recall also that the integral was explained there as an antiderivative; the function integrated (the integrand) was thus the derivative of the integral.

It might be helpful to consider a given function in this way. Being the derivative of the integral, it must express the rate of increase in the area (the value of the integral). Therefore, if the function is constant, if it graphs a horizontal line, the rate at which the area increases must also be constant. We see at once that it is, since the region will be rectangular. Similar arguements prevail when the value of the function increases or decreases --- the area increases faster or slower, respectively.



AS x INCREASES, THE AREA UNDER THE LINE INCREASES AT A CONSTANT RATE, 5.
$A = 5 \cdot x$



AS x INCREASES, THE AREA INCREASES AT AN INCREASING RATE.

Obviously, then, one method of computing the integral would be to use antiderivatives. (This was done in chapter 4.) But we might also divide the area under the curve into rectangles to find a value. In chapter 4 we approximated this value with rectangles from the x-axis rising to the graph of the function. We also noted that if the rectangles became narrower and narrower, the approximation would become better and better; if this process is carried to its limit we obtain a true value for the area -- and the integral.

We shall take up this latter method first, and in order to take the limit we will use summation notation, to which we now turn our attention.

2. Summation notation is a kind of mathematical shorthand to represent a sum of certain members of a series. The notation consists of the Greek sigma, the generalized term, and two indices of summation:

$$\sum_{i=1}^{5} a_i$$

The expression above would be read, "The sum from 1 to 5 of a sub i." It represents the sum of five terms:

$$\sum_{i=1}^{5} a_i = a_1 + a_2 + a_3 + a_4 + a_5$$

The 'i' is a dummy symbol, used only in the single expression, the index of summation. Any letter could be used. The top and bottom numbers are the greatest and smallest values of this index. The value of the index is increased by increments of one.

To use this notation effectively, we must know more about it. Consider c=any constant and d_j any kind of term. Then

$$\sum_{j=1}^{3} cd_j = cd_1+cd_2+cd_3$$
$$= c(d_1+d_2+d_3)$$
$$= c\sum_{j=1}^{3} d_j$$

Similar arguements show that the general case is true:

$$\sum_{j=a}^{n} cd_j = c\sum_{j=a}^{n} d_j$$

We can also show that

$$\sum_{k=x}^{n} (a_x+b_x) = \sum_{k=x}^{n} a_x + \sum_{k=x}^{n} b_x$$

For simplicity, we define

$$\sum_{m=1}^{n} 1 = n = 1+1+1+\ldots+1 \quad \text{(with n terms)}$$

Now we can use a mathematician's trick to obtain a value for the summation from 1 to n of k; the trick is to go both forwards and backwards.

$$\sum_{k=1}^{n} k = 1+2+3+\ldots+n$$

Backwards,

$$n+(n-1)+\ldots+2+1 = \sum_{k=1}^{n} (n+1-k)$$

Adding these will, of course, give us twice the sum we want; this double sum is

$$\sum_{k=1}^{n} k + \sum_{k=1}^{n} (n+1-k) = \sum_{k=1}^{n} (n+1)$$
$$= (n+1)\sum_{k=1}^{n} 1 \qquad \text{(n+1 is a constant)}$$
$$= (n+1)n$$

Since this is twice the value we seek, we multiply by $\frac{1}{2}$ to get

$$\sum_{k=1}^{n} k = \tfrac{1}{2}n(n+1)$$

The next formula is for a summation with k^2. This is

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n\ (n+1)\ (2n+1)$$

It is found by using an unreasonably obscure trick: Take the fact

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1$$

Then say that (therefore) the sum from 1 to n of the Left Member must equal the sum from 1 to n of the Right Member. The equation thus formed, if worked out properly, will eventually result in the formula above.

Now that we have a method that works (once), we try it again; the result is that

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

This one, of course, uses $(k+1)^4 - k^4$. This method also works for summations of k to higher powers.

3. Now back to finding areas. The area under the curve can be divided by vertical lines into areas easily approximated by rectangles. The width of the rectangles is determined arbitrarily by the width of the intervals along the x-axis. The height is determined by the value of the function at some value of x in the interval. It is clear that as the number of intervals increases, and the width of each decreases, the relative widths become less important; whether or not the intervals are equal is unimportant as the limit is approached.

It can also be seen that the placement of the value of x from which the height is taken for the rectangle is not important. The choice is constantly being narrowed and the possible values of the function come closer and closer together. If you would like to see a discussion of this in terms of greatest and least possible areas approaching the same limit, refer to your text; the matter is discussed at great lengthon pages 206-211, also 200-203.

To actually use this information, we set up a summation from a (or somewhere) to n of the rectangles which approximate the area we want. We then take a limit as n approaches infinity and the widths of the rectangles gets small. Precisely and formally stated the definition is quite complex; the simpler but less precise method alluded to at the beginning of this paragraph will be explained with an example following the formal definition. Note that (in accord with the arguments of the preceding paragraphs) the intervals are all taken as the same length and the values of the function at the ends of the intervals.*

*The definition in fully generalized form is on page 214 of the text.

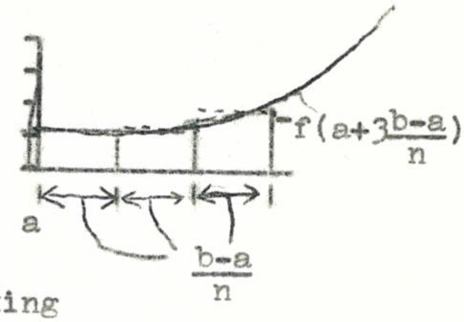
A function is integrable on an interval $[a,b]$ if there exists a number A, the definate integral, with the following property:

For each $e > 0$ there is a corresponding $w > 0$ such that

$$\left| \sum_{k=1}^{n} f\left(a + k\cdot\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right) - A \right| < e$$

for every $\frac{b-a}{n} < w$.

The term $\frac{b-a}{n}$ is simply the width of each rectangle. The value of the function is taken at the end of each rectangle -- a, plus k widths.



As mentioned before, as a practical matter (when using 'consistent' rectangles) we can find A, the definite integral, by taking the limit as $n \to \infty$ of the summation expression. For example,

$$f(x) = x+1 \qquad a=0,$$

$$\sum_{k=1}^{n} f\left(0+k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right) = \frac{b}{n}\sum_{k=1}^{n} f\left(k\frac{b}{n}\right)$$

$$= \frac{b}{n}\sum_{k=1}^{n} k\frac{b}{n} + 1$$

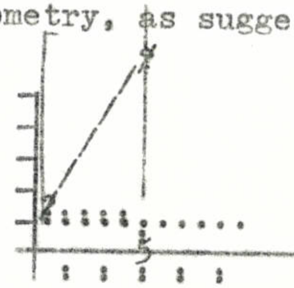
$$= \frac{b}{n}\left(\frac{b}{n}\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right)$$

$$= \frac{b^2}{n^2}\,\tfrac{1}{2}(n^2+n) + \frac{b}{n}n)$$

We now take the limit as $n \to \infty$.

$$\lim_{n\to\infty} \frac{b^2}{2} + \frac{b^2}{2n} + b = \frac{b^2}{2} + b = A$$

Of course, if we left a in the expression we could find an area bounded on the left by some line other than x=0. This result can be checked with geometry, as suggested by the figure:



Triangle is 45° right
add redtangle's area

4. We now return to the antiderivative idea. Since every antiderivative requires the appending of a constant (possibly zero), there are an infinite number of antiderivatives for each function. It seems probable, however, that the only difference between two antiderivatives of a function would be the value of the constant. It is this idea we now wish to prove:

If $F_1(x)$ and $F_2(x)$ are different antiderivatives of the function f in some interval, then $F_1(x) = F_2(x)$ + (some constant) in that interval.

Take $G(x) = F_1(x) - F_2(x)$

Then $G' = F_1' - F_2'$ but since F_1 and F_2 are antiderivatives of the same function, their derivatives must both equal f.

Thus, $G' = 0$

By the Theorum of the Mean (#4 of chapt. 7)

$G(x_1) - G(x_2) = G'(\bar{x})(x_1 - x_2)$ where all x are in the interval in question and $x_2 \leq \bar{x} \leq x_1$.

Since $G'(x)=0$, $G(x_1) = G(x_2)$ for all x_1 and x_2 in the interval. In other words, G is constant, and the theorum is proved.

To continue our discussion, we should reconsider the expression for the integral:

$$\int_a^b f(x)\,dx$$

This is read, "The definite integral from a to b of f." The letter x is the variable of integration. Like the index of summation, this is a dummy variable. The function (here denoted as f) is the integrand. The a and b are the lower and upper limits of integration.

Suppose F(x) is an antiderivative of f (in [a,b]). Recalling that the integral is also an antiderivative, there must be a constant, c, so that

$$\int_a^x f(t)dt = F(x) + c$$

for all x in [a,b].

We can solve for c by using the case, included above, when x=a. Then

$$0 = F(a)+c$$

Or

$$c = -F(a)$$

Substituting the constant back into the equation yields

$$\int_a^x f(t)dt = F(x) - F(a)$$

We now, finally, have a way to compute any integral for which we know an antiderivative. This computation is expedited by the use of still another symbol. We write

$$F(x)\Big]_a^b \text{ or } \Big[F(x)\Big]_a^b \quad \text{for} \quad F(b)-F(a).$$

For example,

$$\int_0^2 (2x+1)dx = x^2+x\Big]_0^2$$
$$= 6 - 0$$
$$= 6$$

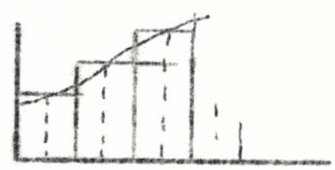
With relatively limited knowledge of antiderivatives, there may be times when neither the antiderivative nor the summation method will be very efficient. Until we know more, we may wish to approximate by rectangles; one way is to use the Midpoint Rule. This uses the same method as implied by the definition; instead of having n=an infinite number, we use a finite number of intervals (each equal to the others). The value of the function is taken at the midpoint of each interval.

MIDPOINT RULE

$$\sum_{k=1}^{n} f(\bar{x})\frac{b-a}{n}$$

where $\bar{x} = a + (k-\frac{1}{2})\frac{b-a}{n}$

which is the midpoint of the k^{th} interval.



Before finding any real use for our ability, we will formalize and further our purely mathematical studies of the integral.

5. The formal definition of the definite integral has already been given. The following definitions are concerned with area and reveal several properties of the integral:

AREA OF A RECTANGLE is its length times its width.

AREA of a region which is the UNION OF NON-OVERLAPPING RECTANGLES is the sum of the areas of these rectangles.

AREA BOUNDED BY $x=a$, $x=b$, $y=0$ AND $y=f(x)$, where f is integrable in $[a,b]$ and positive when $a<x<b$, is the value of

$$\int_a^b f(x)dx.$$

AREA bounded by $x=a$, $x=b$, $y=0$, and $y=f(x)$, where f is integrable $a\leq x\leq b$ but NEGATIVE $a<x<c$, is

$$-\int_a^b f(x)dx.$$

AREA BOUNDED BY $y=c$, $y=d$, $x=0$, AND $x=g(y)$ is

$\int_a^b g(y)dy$, if $g>0$ when $c<y<d$,

$-\int_c^d g(y)dy$, if $g<0$ when $c<y<d$, where g is integrable in the interval $c\leq y\leq d$.

ANY AREA WHICH CAN BE DIVIDED INTO A FINITE NUMBER OF REGIONS

FOR WHICH AREAS ARE KNOWN IS THE SUM OF THOSE AREAS.

The fact that all usable methods will yield the same result will not be proven here.

6. Up to now we have tacitly assumed that integrals exist -- but we have also only used continuous functions as examples. We will now present some theorums partially answering the question, "What theorums are integrable?"

The first two theorums can be proven by using greatest and least approximations; this is done in the text on pages 218 and 216. (They are presented in 'reverse' order.)

THEORUM 1: If f(x) is continuous in $[a,b]$ it is integrable there.

THEORUM 2: If f(x) is non-increasing or non-decreasing in $[a,b]$ it is integrable in a,b .

This follows from #1.

THEORUM 3: If f(x) is integrable in $[a,b]$, and c is constant,

$$\int_a^b c\, f(x)dx = c\int_a^b f(x)dx.$$

According to our definition,

$$\int_a^b cf(x)dx = \lim_{n\to} \sum_{k=1}^{n} \left[c\, f(a + \frac{b-a}{n}) \frac{b-a}{n}\right].$$

According to the rules for summation, we may take out the constant c, leaving the expression of the definition of the desired integral, times c.

THEORUM 4: If f(x) and g(x) are integrable in $[a,b]$,

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

The proof follows the pattern of #3.

THEORUM 5: If f(x) is integrable in $[a,b]$, it is bounded there; there exists an m and an M such that $m \leq f(x) \leq M$ for $a \leq x \leq b$.

This can be seen intuitively by considering the are under the curve as the function becomes infinite.

THEORUM 6: If f(x) is integrable in $[a,b]$ and $m \leq f(x) \leq M$ for $a \leq x \leq b$,

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

We multiply by $\frac{b-a}{n}$ in one of our conditions to get

$$m(\frac{b-a}{n}) \leq f(x)(\frac{b-a}{n}) \leq M(\frac{b-a}{n}).$$ [Here x may be taken as $(a+k\frac{b-a}{n})$]

We then take the summation from 1 to n:

$$\sum_{k=1}^{n} m(\frac{b-a}{n}) \leq \sum_{k=1}^{n} f(x)(\frac{b-a}{n}) \leq \sum_{k=1}^{n} M(\frac{b-a}{n})$$

The extreme members can be worked out, giving

$$m(b-a) \leq \sum_{k=1}^{n} f(x)\left(\frac{b-a}{n}\right) \leq M(b-a).$$

We finally take the limit as $n \to \infty$. The extreme members are constants, and thus are not affected; the middle member becomes the integral.

THEORUM 7: If f and g are integrable in $[a,b]$ and $f(x) \leq g(x)$ for all $a \leq x \leq b$,

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

We define $F(x)=g(x)-f(x)$. $-f(x)$ is integrable (#3; $c=-1$) and $g(x)$ is integrable; thus $F(x)$ is also integrable (#4). Further, $0 \leq g(x)-f(x)=F(x)$, so we may choose $m=0$ to substitute into the equation of #6:

$$m(b-a) = 0 \leq \int_a^b F(x)dx$$

Thus

$$\int_a^b (g(x)-f(x))\, dx \geq 0.$$

By theorum #4, this is

$$\int_a^b g(x)dx - \int_a^b f(x)dx \geq 0,$$

or

$$\int_a^b g(x)dx \geq \int_a^b f(x)dx.$$

THEORUM 8: If $f(x)$ is continuous in $[a,b]$ and $a \leq c \leq b$,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

This idea follows from the quite obvious one that if you add up some terms, then the others, and then add the two sums, you will obtain the same answer as adding all the terms the first time. This is then applied the the summations which in turn (when the limit is taken) yield the integrals.

COROLLARY TO #8: Theorums 8 remains true for any a, b, and c such that the integrals exist.

The proof follows from the supplementary definitions following.

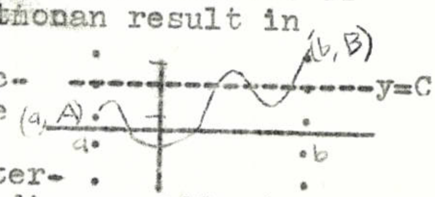
DEFINITION: If $a \geq b$, then $\int_a^b f(x)dx = -\int_b^a f(x)dx.$

This also means that $\int_a^a f(x)dx = 0.$

7. Our formal study of the definite integral now takes us to the idea which we have accepted since chapter four - that the integral is a derivative. We must first discuss two properties used in the proof:

THEORUM 10:* INTERMEDIATE VALUE THEORUM: If f(x) is continuous on [a,b], and f(a)=A and f(b)=B, then for C between A and B there exists a c between a and b such that f(c)=C.

The proof will not be given here, but we will consider the validity. Since the function is continuous on the interval, and the interval is closed, no value of the function can result in the function being infinite. Thus it is bounded. The function must take on every value between A and B, since it is continuous, and thus will intersect y=C at least once. The diagram illustrates this.



Obviously this may not be true if the function is not continuous, including cases where it becomes infinite, because the function may never take on a value which we might choose.

THEORUM 11: THEORUM OF THE MEAN FOR INTEGRALS: If f is continuous on [a,b], there is a number c between a and b such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

By #5 (and the Extreme Value Theorum of chapter 3) we know that there are bounds, m and M. By #6 we have

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

This statement implies that there is one number (between m and M) which will give the exact value of the integral, that is a D such that

$$\int_a^b f(x)dx = D(b-a).$$

Since the Extreme Value Theorum tells us that we can choose values of x so that $f(x_0)=m$ and $f(x_1)=M$ (with the x values in [a,b]), we can employ #10 to say that the value D is taken on by the function in the interval.

Geometrically, this is akin to saying that there is an average value of the function and that this average value can be used to construct a rectangle with the same area as the area under the curve. It would be very difficult to find such a value, but we will not have to to use it in our next proof.

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* The careful reader will note that we skipped 9. The theorum called "9" in the text has already been presented; this keeps the numbering the same.

THEORUM 12: FUNDAMENTAL THEORUM OF CALCULUS: The function f is continuous on $[a,b]$ and c is a number in this interval. We define

$$F(x) = \int_c^x f(t)dt$$

for each x in the interval. Then

$$F'(x) = f(x).$$

So long as x+h is within the interval, #8 tells us that

$$\int_c^{x+h} f(t)dt = \int_c^x f(t)dt + \int_x^{x+h} f(t)dt.$$

We have defined F so that this is the same as

$$F(x+h) = F(x) + \int_x^{x+h} f(t)dt.$$

We now apply the Theorum of the Mean for Integrals to obtain

$$\int_x^{x+h} f(t)dt = f(c)\cdot h \qquad \text{(c in the interval)}$$

and

$$f(c) = \frac{F(x+h) - F(x)}{h}.$$

By the definition of continuity we can say that

$$\lim_{c\to x} f(c) = f(x).$$

At the same time, as $h\to 0$ $x+h \to x$ and (since c must be between x and x+h) $c\to x$.

By substitution for f(c) and the previous statement we write

$$\lim_{h-0}\frac{F(x+h) - F(x)}{h} = f(x)$$

or

$$F'(x) = f(x).$$

The hypotheses may be less restrictive; such theorums are proved in more advanced courses.

The Fundamental Theorum implies that every (continuous) function has an antiderivative. It also means that differentiation and integration are inverse processes, like addition and subtraction or multiplication and division (except division by zero). Integration and differentiation are inverse only when some extra condition is included; as we have proved it, the condition is that "f is continuous."

You will notice that this theorum proves what we have been saying all along. Being proved does have advantages, however. The careful student will notice that we have proved this without begging the question; that is, without using the intuitive argument of antiderivatives we used early in the chapter.

8. We do not have a systematic method of finding antiderivatives, and when expressions become long it is often very difficult to find any. In this section we will find a way to calculate some of these complex expressions. It is obvious that every formula for finding derivatives also helps to find antiderivatives, but often direct application of these equations to a given problem is not possible. In this section we will be looking at the application of the chain rule this way.

Although our purpose is to simplify the use of the chain rule in finding antiderivatives, we begin by introducing a new symbol. We have been using the definate integral and we have said that this is a certain antiderivative (see part 4 and part 7, theorum #12). If the limits of integration are not specified, however, we don't know which antiderivative to use. It is only natural therefore that the following symbol be called the "indefinate integral" and represent the general antiderivative of the function (f):

$$\int f(x)\,dx$$

A third synonym for the antiderivative is "primative function." We have this symbol in this chapter solely for the purpose of making a theorum easy to prove. We use the indefinate integral in the following theorum, really a lemma:

THEORUM: Within a closed interval $[a,b]$: Let h be a function of x with a continuous derivative and let g be a continuous function of u, where $u=h(x)$. Then

$$\int g(h(x))\,h'(x)\,dx = \int g(u)\,du.$$

From the chain rule we know that if we define $G(x) = \int g(x)$,

$$G'(h(x)) = g(h(x))\,h'(x).$$

In differentials we would write

$$\frac{dG(h(x))}{dx} = g(h(x))h'(x).$$

Similarly, $dG(u) = g(h(x))h'(x$
or $dG(u) = g(u)h'(x)dx$, since $u=h(x)$.

These equations tell us that the antiderivative of $g(u)du$ is equivalent to the antiderivative of $g(h(x))dx$. (Besides substitution, this conclusion requires theorum 12 which implies that integration and differentiation are inverse processes.)

The careful student will note that the preceding discussion explains the inclusion of the "dx" notation. With the functions used in the proof, we pretend the expressions are for the differentials
Thus: $dG(h(x)) = g(h(x))h'(x)dx$ and $dG(u) = g(u)du$.
But $du = h'(x)dx$. Thus when using this and the following theorum the "dx" notation serves as an aid the to memory.

~~When finding definate integrals, the same idea is useful:~~
~~THEORUM: Under the same conditions as above:~~

9. There is some direct application of the theorem for finding indefinite integrals to our work with definite integrals since we have shown that definite integrals are antiderivatives. However, most applications to definite integrals make use of the following theorum, which is based on the preceding one:

THEOREM: Under the same hypotheses as in the preceding theorem:

$$\int_a^b g(h(x))h'(x)dx = \int_{h(a)}^{h(b)} g(u)du. \qquad (u=h(x))$$

The preceding theorem means that if G(u) is an antiderivative of g(u), then G(h(x)) is the antiderivative for g(h(x))h'(x). We then have from the Fundamental Theorem of Calculus

$$\int_a^b g(h(x))h'(x)dx = G(h(x))\Big|_{x=a}^{x=b}$$
$$= G(h(b)) - G(h(a)).$$

We also have

$$\int_{h(a)}^{h(b)} g(u)du = G(u)\Big|_{u=h(a)}^{u=h(b)}$$
$$= G(h(b)) - G(h(a)).$$

These equations provide proof of the theorum.

The use of change of variable is illustrated by the following examples. The first method makes use of the theorem of the preceding section and the second requires the theorem of this section. For most problems the second method is easier.

10. Area between curves is the sum or difference of the areas from the curves to the axis. A thorough and illustrated discussion of this is given in your book.

Refer to 8. AREA BETWEEN CURVES; p. 238.

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11. The integral is useful for many things besides finding area. In most cases the applicability of the integral is proven by using the definition. As a student, you may wish it hard to understand how the integral can be used in any of these. If so, you may wish to try to apply the intuitive reasoning we used in the first section of this chapter in regard to area. For example, is it logical to say that the function expressing motion will be the derivative of the function expressing the work done? If you are convinced of the truth of this, the use of the definate integral in work problems is logical.

For the remainder of the chapter you will be working in the textbook. The authors of your text have used a slightly different symbolisms: Instead of having divisions along the x-axis uniform and equal to $\frac{b-a}{n}$, they use the more general x_i. As a general value for x (between x_i and x_{i+1}) they chose (   ), and to represent the difference $x_{i+1}-x_i$, that is, the width of the interval, they use $\Delta_i x$.

12. Begin work on p. 244: 9. WORK.